Maximizing Spectral Efficiency for Energy Harvesting-Aware WBAN

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Abstract—In this paper, we investigate the spectral efficiency of a communication link in a wireless body area network (WBAN) capable of harvesting energy from the environment. We consider two scenarios for the transmission which are single- and dual-hop, and achieve the power management policy for each scenario. In the first scenario, the aim is to maximize the link’s spectral efficiency over $N$ time slots subject to the battery capacity, energy harvesting (EH) constraint, and WBAN limitations including power and outage probability. In the second scenario, a decode-and-forward relay node is considered, and a spectral efficiency optimization problem with constraints similar to the first scenario is evaluated. In addition, since the channel distribution information (CDI) is available at the transmitters, the lower and upper bounds of the average spectral efficiency are also derived in both scenarios. Finally, numerical results corroborate the analytical results.

Index Terms—Dual-hop, energy harvesting, lower bound, outage probability, single-hop, spectral efficiency, upper bound, WBAN.

I. INTRODUCTION

Recent advances in wireless sensor networks (WSNs) and electronic equipments have developed tiny, smart, and low-power sensors. The main role of these sensors is to collect data from the environment and send it to an external access point (AP). Generally, a wireless body area network (WBAN) is a dynamic network which consists a number of sensors. The location of these sensors can be over or under the skin, i.e. on-body or in-body, respectively. In-body area networks provide communication between implanted sensors, whereas on-body area networks do the same job among wearable sensors [1]–[3], and additionally, these nodes connect to each other through a wireless communication channel.

WBANs provide some useful applications, such as medical and personal health care, entertainment, sports, fitness, etc. Various applications of WBAN impose different constraints on the network design, including data rate, bit error rate (BER), and power consumption [4]. On the other hand, the outage probability of the network is a critical quality of service (QoS) requirement. Therefore, the data rate control is an important challenge in the WBAN design.

Regular WSN and WBAN infrastructures are provided with fixed energy supply. Since the charging process of sensor nodes is not preferable and battery replacement of sensors in some cases, especially in in-body area networks, may not be a viable choice, the idea of energy harvesting (EH) is an appropriate approach for the energy provision. In contrast to conventional communication systems which are only subject to energy constraints, transmitters with EH capabilities are subject to EH constraints, as well [5]–[7].

A. Technical Literature Review

In [5], the minimization problem of the average outage probability over $N$ EH blocks was investigated. The authors assumed that the channel state information (CSI) is perfectly known at the receiver, while only the channel distribution information (CDI) is known at the transmitter. The problem of energy allocation over a finite horizon was studied in [6]. In order to maximize the spectral efficiency, dynamic programming and convex optimization techniques were used. The authors in [8] considered an optimization problem of a node-to-node communication with EH capabilities in a wireless fading channel, where a directional water-filling algorithm is proposed. In [9], power allocation schemes were proposed for EH relay systems, where off-line and on-line power allocations are formulated by convex optimization and dynamic programming methods, respectively. The authors in [10] formulated the throughput maximization problem with variable time slot duration for a dual-hop link with EH constraints. The propagation channel of WBAN was discussed in [11], [12] and channel characteristics, path-loss parameters, and small-scale fading models were extracted both from the measurements and simulation results. The throughput and delay performance of the IEEE 802.15.6 standard was presented in [13], where the authors derived some equations for the maximum throughput and minimum delay. Finally, in [14], the optimal transmission power and the location of the sink are obtained by solving an optimization problem.

B. Motivation and Contribution

In this paper, a node-to-node communication link in an on-body area network with the EH capability is considered. By jointly considering the QoS requirements of the network and characteristics of the channel between sensor nodes and EH constraints, the problem of power allocation is formulated as
a spectral efficiency maximization problem, subject to the outage probability, EH constraints, and buffer capacity limitation. To the best of our knowledge, none of the previous works focus were on spectral efficiency in WBAN single- and dual-hop links. Moreover, the outage constraint which is critical in health-related applications has not been studied in the WBAN literature so far. By solving the optimization problem, optimal transmission powers are obtained. The problem is formulated for both single and dual-hop links. It is also assumed that the CDI and perfect CSI are available at the transmitter and the receiver, respectively. Since only the CDI is available at the transmitter, the lower and upper bounds for the average spectral efficiency is calculated by exploiting the Jensen’s inequality.

Designing nodes which are capable of harvesting energy from the surrounding area has attracted researchers’ attention in recent years (see, e.g., [8] and [9]). Furthermore, spectral efficiency plays an important role in QoS provisioning of WBANs. Therefore, studying EH-aware WBANs can improve the performance of these networks and achieve a power management policy leading a better power consumption in comparison to the conventional case. The theoretical results of this paper can be employed in design of wireless body sensors with different data rate required applications. A further implication of this analysis is that it can help to design an EH-aware network with an optimal power management which is an important issue in in-body and on-body area networks.

The rest of the paper is organized as follows. In Section II, the channel and network models are described in separate subsections. The power allocation problem is formulated in Section III. Section IV is allocated to lower and upper bounds of the problems for both single- and dual-hop scenarios. Simulation results are presented in Section V. Finally, the paper is concluded in Section VI.

1Throughout this work, throughput and spectral efficiency are used interchangeably.
could be different from point to point. Generally, in-body and on-body communication channels work in the range of 402 – 405 MHz [19].

The received signal in the $i$-th time slot is given by $y_i = h_i x_i + n_i$, where $h_i$, $x_i$, and $n_i$ are the channel gain of the link between the transmitter and the receiver, the transmitted signal, and the additive white Gaussian noise (AWGN) with zero-mean and variance $N_0$, respectively. As described earlier, statistical distribution of the fading channel is assumed to be log-normal.

B. Network Model

We consider two different scenarios for data transmission. These scenarios are single and dual-hop links in a WBAN. In the sequel, we will discuss these two scenarios in detail.

1) Single-Hop Link: According to the application type, the transmitting nodes may be placed on the body surface or implanted in the body, and their typical positions may be arm, back, leg, head, etc. We consider a single-hop WBAN as depicted in Fig. 3(a). We assume that links between the sink and the collecting nodes have EH capabilities, i.e., each node harvests energy from the environment during each time slot. The transmitter has data and energy buffers, which the former stores the arriving data packets and the latter saves the harvested energy.

We applied time division multiple access (TDMA) scheme for the packet collision avoidance. As shown in Fig. 3(b), we assume that there are $N$ time slots, each has a duration of $T$ seconds. The harvested energy arrives during a time slot and is stored in the battery and then can be consumed in the next time slot. The time series $t_0, \ldots, t_N$, with $t_0 = 0$, shows the end of the time slots and the energy series $E_0, \ldots, E_{N-1}$ describes the harvested energy in the previous time slot. Since TDMA is applied, the duration of all time slots are the same and equals to $T$, i.e., $t_i - t_{i-1} = T$.

The received signal in the $i$-th time slot is given by

$$y_i = h_i \sqrt{p_i} x_i + n_i,$$  \hspace{1cm}  \hspace{1cm} (2)

where $p_i$, $h_i$, $x_i$, and $n_i$ represent the transmitting power, the channel gains of the link between the transmitter and the receiver, the transmitted signal, and the AWGN with zero-mean and variance $N_0$, respectively. As described earlier, statistical distribution of the fading channel is assumed to be log-normal.

For the EH scheme, two models can be considered. Non-causal energy state information (ESI) and causal ESI [5]. In the first model, it is assumed that all $E_i$’s are known at the transmitter and in the second one, at the $i$-th time slot, $i = 1, 2, \ldots, N$, only $E_0, \ldots, E_{i-1}$ are known at the transmitter. In this paper, we consider the spectral efficiency maximization problem in a WBAN with EH capabilities. The applied model is non-causal ESI in which an off-line power management policy is obtained, i.e., the transmitter knows all the energy arrival instants and amounts as well as the channel gains for the time slots in advance in $t = 0$.

2) Dual-Hop Link: Now, we focus on the dual-hop link data transmission system model in a WBAN. In addition to the proposed model in the single-hop data transmission, another node acts as a relay between the transmitter and the receiver without line-of-sight (LOS). It is assumed that the relay node has data and energy buffer similar to the transmitter node. It’s worth mentioning that the capacity of data buffer of both the transmitter and relay nodes can be either finite or infinite. Fig. 3(a) depicts a dual-hop WBAN link. Similar to the single-hop mode, the TDMA protocol is applied for the packet collision avoidance. Moreover, it is assumed that the transmitter node and the relay do not transmit data simultaneously. Therefore, we consider two transmission phase which the transmitter or the source node sends its data to the relay node in the first phase, and in the second phase, the relay node sends the received data to the receiver or the destination node. Although a decode-and-forward (DF) relay has a higher complexity than an amplify-and-forward (AF) relay due to its full processing capability, as shown in [21], single-antenna multi-hop Rayleigh-fading relay channels with DF protocol achieve higher ergodic capacity in comparison to the AF relaying. In WBANs applications in which we are facing with power constraints in some cases, a protocol with higher capacity is preferable. Here, we assume that the relay employs the DF protocol in which the data is decoded and re-encoded before retransmission.

The transmitted signal experiences additive noise and fading effect in both channels, i.e. source-relay and relay-destination channels. Mathematically speaking, the received signals at the relay and final receiver nodes can be expressed as

$$y_{r,i} = h_{sr,i} \sqrt{p_i} x_i + n_{sr,i}, \quad i = 1, 2, \ldots, N,$$  \hspace{1cm}  \hspace{1cm} (3)

$$y_{d,i} = h_{rd,i} \sqrt{p_i} \hat{x}_i + n_{r,i}, \quad i = 1, 2, \ldots, N,$$  \hspace{1cm}  \hspace{1cm} (4)

where $x_i$, $\hat{x}_i$, $h_{sr,i}$, $h_{rd,i}$ are the transmitted signal from the source to the relay, the decoded signal, the source-relay, and
the relay-destination channel gain, respectively. In addition, \( n_{s,i} \) and \( n_{r,j} \) are independent and identically distributed Gaussian noise random variables (RVs) with zero-mean and variance \( N_0 \). The series \( E_{s,0}, \ldots, E_{s,N-1} \) and \( E_{r,0}, \ldots, E_{r,N-1} \) describe the harvested energy in the passed time slots for the source and the rely, respectively.

### III. Optimization Problem Formulation

In this section, the problem formulation for calculating the off-line power management policy is investigated for both scenarios. The goal is to allocate a power vector for all time slots in order to maximize the spectral efficiency of the link with EH constraints.

#### A. Single-hop Link

The channel is assumed to experience additive white Gaussian noise (AWGN), and thus, the achievable spectral efficiency for each node in each time slot is given by [22]

\[
\eta_{SE,i} = \frac{R_i}{W} = \log_2 \left( 1 + \frac{|h_i|^2 p_i}{N_0} \right),
\]

where \( W \) and \( p_i \) denote the bandwidth of the channel and the transmission power of a node in the \( i \)-th time slot, respectively. Note that the fraction \( \frac{|h_i|^2 p_i}{N_0} \) is the signal-to-noise-ratio (SNR) at the receiver. In [5], \( R_i \) is the number of bits that can be transmitted by the source over a channel whose bandwidth is \( W \) in the \( i \)-th time slot. Here, we consider two optimization problems. First, we study the case without a battery constraint. Then, we propose a solution for the problem which is subjected under battery constraints.

1) **Infinite Battery Capacity**: We assume an infinite battery capacity, which gives the maximum accessible spectral efficiency. Since there is no constraint in the problem, the solution is found to be an upper-bound for the problem of finite battery capacity. The power management policy is denoted by \( P^* = [p_1, p_2, \ldots, p_N] \). For the first problem, we have two constraints for the power values and one constraint due to the energy causality. In other words, the transmitter cannot use an energy which has not arrived yet and it can only consume the stored energy in the battery. Mathematically speaking, we have

\[
\sum_{i=1}^{m} T p_i \leq \sum_{i=1}^{m} E_i - 1, \quad m = 1, 2, \ldots, N. \tag{6}
\]

Since the transmitter should consume all the stored energy in the battery in the last time slot, the inequality in (6) is changed into an equality for \( m = N \).

In addition to the above constraints, the outage probability is another constraint which should be considered to guarantee the QoS requirement. The outage probability can be written as

\[
P_{out} = \Pr (\text{SNR} \leq \beta) = \Pr \left( |h_i|^2 \leq \frac{\beta N_0}{p_i} \right) = \frac{1}{2} + \frac{1}{2} \text{erf} \left( \frac{\ln(\beta N_0) - \ln p_i - 2\mu}{2\sqrt{2}\sigma} \right), \tag{7}
\]

where \( \mu \) and \( \sigma \) are log-normal parameters and \( h_i, \beta, \) and \( \text{erf}(.) \) denote channel gain, SNR threshold, and the standard error function, respectively.

Hence, the problem can now be formulated as

\[
\max_{p_i} \sum_{i=1}^{N} \mathbb{E} \left[ \log_2 \left( 1 + \frac{|h_i|^2 p_i}{N_0} \right) \right] \tag{8}
\]

s.t. \( \sum_{i=1}^{m} p_i \leq \sum_{i=1}^{m} E_i - 1, \quad m = 1, 2, \ldots, N, \)

\[
P_{out} = \frac{1}{2} + \frac{1}{2} \text{erf} \left( \frac{\ln(\beta N_0) - \ln p_i - 2\mu}{2\sqrt{2}\sigma} \right) \leq \chi, \]

\[
p_i \leq P_{max}. \tag{9}
\]

where \( \chi \) stands for outage probability threshold. By manipulating the outage constraint in (9), it can be reduced to a minimum power consumption on \( p_i \) which for simplicity we name it \( P_{min} \). Thus, the constraint is modified as

\[
p_i \geq \beta N_0 \exp \left( -2\mu - 2\sqrt{2} \sigma \text{erf}^{-1}(2\chi - 1) \right) = P_{min}. \tag{9}
\]

Since only the CDI is available for the problem stated in (8), the average spectral efficiency of the link should be considered. In other words, the expectation of the problem over \( h_i \) should be calculated. To pursue this aim, according to [23, p. 2635], a simple way to eliminate the expectation is to calculate the optimal problem for a specific value of the RV for a relatively large number of iterations and average the achieved values of optimal spectral efficiency. To put it in another way, by the Law of Large Numbers, as \( N \rightarrow \infty \) the optimal objective value and the average of the optimal values of the deterministic problem will converge.

The objective function in (8) is sum of log(.) functions which are concave themselves. Thus, the objective function is concave in the power variables. Since the constraint set is composed of linear constraints, the problem in (8) is a convex optimization problem and can be solved with convex optimization methods [24].

**Proposition 1.** The optimal value of the problem in (8) is

\[
p_i^* = \left[ \frac{\alpha}{T \sum_{j=1}^{N} \lambda_j - \frac{N_0}{|h_i|^2}} \right] p_{max} \tag{10}
\]

where \( \frac{x}{P_{min}} \) denotes a function which gives \( x \) clips to be between \( P_{min} \) and \( P_{max} \).

**Proof.** The proof is given in Appendix A. \( \square \)

2) **Finite Battery Capacity**: In each time slot, summation of the stored energy in the battery and the arrival energy in the current time slot should not exceed the battery capacity, \( B_{max} \). This constraint can be formulated as

\[
\sum_{i=1}^{m+1} E_{i-1} - \sum_{i=1}^{m} T p_i \leq B_{max}, \quad m = 1, 2, \ldots, N - 1. \tag{11}
\]

Now, we modify the problem stated in (8) as follows and solve it similar to infinite battery capacity case.
Proof. The proof is given in Appendix A.

The solution of (10) and (13) is found by conventional direction water-filling algorithm, respectively. Both \( \alpha(T \sum_{j=1}^{N} \lambda_j)^{-1} \) and \( \alpha(T (\sum_{j=1}^{N-1} \nu_j))^{-1} \) are water levels according to the water-filling algorithm.

B. Dual-hop Link

In this subsection, we consider the extension of the described system model in Subsection III-A and formulate it to maximize the end-to-end spectral efficiency according to EH and outage probability constraints. The main goal is to calculate the power management policy in such a way that the summation of spectral efficiency of the link in the \( N \) time slots would be maximized. As it was mentioned before, the relay employs DF protocol and the achievable spectral efficiency in this link is given by (25)

\[
\eta_{SE_i} = \frac{R_i}{W} = \frac{1}{2} \min \left( \log_2 (1 + \xi_{sr,i}), \log_2 (1 + \xi_{rd,i}) \right).
\]

According to the fact that the \( \log_2(.) \) is an ascending function, (14) can be rewritten as

\[
\eta_{SE_i} = \frac{1}{2} \log_2 \left( 1 + \min (\xi_{sr,i}, \xi_{rd,i}) \right),
\]

where \( \xi_{sr,i} \) and \( \xi_{rd,i} \) denote source-relay and relay-destination links SNR which are equal to \( \frac{p_{sr}^i |h_{sr,i}|^2}{N_0} \) and \( \frac{p_{rd}^i |h_{rd,i}|^2}{N_0} \), respectively. Moreover, the vector representation of the source and destination power consumption are \( P_{sr} = [p_{sr}^1, p_{sr}^2, \ldots, p_{sr}^N] \) and \( P_{rd} = [p_{rd}^1, p_{rd}^2, \ldots, p_{rd}^N] \), respectively.

1) Infinite Battery Capacity: In addition to the fact that the source is able to harvest energy, the relay node is capable to provide its need for energy. Therefore, we can formulate the constraints according to EH as

\[
T \frac{m}{2} \sum_{i=1}^{m} p_{sr}^i \leq \sum_{i=1}^{m} E_{sr}^i, \quad m = 1, 2, \ldots, N, \tag{16}
\]

\[
T \frac{m}{2} \sum_{i=1}^{m} p_{rd}^i \leq \sum_{i=1}^{m} E_{rd}^i, \quad m = 1, 2, \ldots, N, \tag{17}
\]

where \( p_{sr}^i, p_{rd}^i, E_{sr}, \) and \( E_{rd} \) are the source and the relay power consumption, the source and the relay harvested energy, respectively. As it mentioned before, the constraints in (16) and (17) are due to energy causality and these inequalities is changed to equality for the \( N \)-th time slot.

According to the spectral efficiency maximization problem, we face another constraint due to data causality. Generally, the transmitted data in the source-relay link should not be greater than the relay-destination link. The reason is that the relay node does not have any data to transmit intrinsically and just forwards the received data from the source to the destination [9]. Since only the CDI is available, we assume that the expectation of \( \xi_{sr,i} \) and \( \xi_{rd,i} \), which are the links SNR in the \( i \)-th time slot, will be equal, i.e.,

\[
p_{sr}^i \mathbb{E} \left[ |h_{sr,i}|^2 \right] = p_{rd}^i \mathbb{E} \left[ |h_{rd,i}|^2 \right], \tag{18}
\]

where \( |h_{sr,i}|^2 \) and \( |h_{rd,i}|^2 \) obviously have log-normal distribution.

The next constraint is due to outage probability of the source-destination link. As discussed before, we refer to the outage when the SNR at the receiver takes a value below a threshold such as \( \beta \). The outage probability can be calculated as

\[
P_{out} = P_{out}^sr + P_{out}^rd - P_{out}^sr P_{out}^rd, \tag{19}
\]

where \( P_{out}^sr \) and \( P_{out}^rd \) are calculated with the help of the Log-normal cumulative distribution function (CDF).

\[
P_{out}^sr = \frac{1}{2} + \frac{1}{2} \text{erf} \left( \frac{\ln(\beta N_0) - \ln p_{sr}^i - 2\mu_{sr}}{2 \sigma_{sr} \sqrt{2}} \right),
\]

\[
P_{out}^rd = \frac{1}{2} + \frac{1}{2} \text{erf} \left( \frac{\ln(\beta N_0) - \ln p_{rd}^i - 2\mu_{rd}}{2 \sigma_{rd} \sqrt{2}} \right).
\]

In addition to all the considered constraints, there is a limitation on the transmitted power of the source and the relay. In other words, according to WBAN limitations, the transmitted power of each node has to be less than a value named as \( P_{max} \).

Now, we can formulate the cost function and the constraints which we considered above.
\[
\max_{p_i^{sr}, p_i^{rd}} \sum_{i=1}^{N} \frac{1}{2} E_h \left[ \log_2 \left( 1 + \frac{\min\left\{ p_i^{sr} | h_{sr,i} |^2, p_i^{rd} | h_{rd,i} |^2 \right\}}{N_0} \right) \right] \\
\text{s.t.} \quad T \sum_{i=1}^{m} p_i^{sr} \leq \sum_{i=1}^{m} E_{i-1}^s, \quad m = 1, 2, ..., N, \\
T \sum_{i=1}^{m} p_i^{rd} \leq \sum_{i=1}^{m} E_{i-1}^r, \quad m = 1, 2, ..., N, \\
p_i^{sr} E \left[ |h_{sr,i}|^2 \right] = p_i^{rd} E \left[ |h_{rd,i}|^2 \right], \\
P_{out} = P_{out}^s + P_{out}^d - P_{out}^{sr} - P_{out}^{rd} \leq \chi, \\
0 \leq p_i^{sr} \leq P_{max}, \quad 0 \leq p_i^{rd} \leq P_{max}. \quad (20)
\]

As it is assumed that the CDI is known, we can simplify the third constraint in the spectral efficiency optimization problem with the help of log-normal expected value. If \( x \) is supposed to have a log-normal distribution with \( \mu \) and \( \sigma^2 \) parameters, its expected value will be \( e^{\mu+\sigma^2/2} \). Therefore, there will be a relation between the power consumption variables as

\[
p_i^{sr} E \left[ |h_{sr,i}|^2 \right] = p_i^{rd} E \left[ |h_{rd,i}|^2 \right] \Rightarrow p_i^{rd} = \psi p_i^{sr}, \quad (21)
\]

where \( \psi = e^{2(\mu_{sr}-\mu_{rd})+2(\sigma_{sr}^2-\sigma_{rd}^2)} \). In addition, \( (\mu_{sr}, \sigma_{sr}^2) \) and \( (\mu_{rd}, \sigma_{rd}^2) \) are the parameters of source-relay and relay-destination channels. The fourth constraint which is about the outage probability can be rewritten as

\[
P_{out} = P_{out}^{sr} + P_{out}^{rd} - P_{out}^{sr} - P_{out}^{rd} \leq \chi, \quad (22)
\]

\[
erf \left( \frac{\ln \left( \frac{\beta N_0}{\psi p_i^{sr}} \right) - 2 \mu_{sr}}{2 \sigma_{sr} \sqrt{2}} \right) + erf \left( \frac{\ln \left( \frac{\beta N_0}{\psi p_i^{rd}} \right) - 2 \mu_{rd}}{2 \sigma_{rd} \sqrt{2}} \right) \leq \chi. \quad (23)
\]

As the input variable of the error function, \( erf(x) \), takes a value near zero, we can approximate the error function with the Taylor series. By omitting the high order terms, we achieve a simpler inequality for the problem. This inequality is given in (24). Therefore, we can rewrite the spectral efficiency optimization problem with all the simplified inequalities as below.

\[
\max_{p_i^{sr}, p_i^{rd}} \sum_{i=1}^{N} \frac{1}{2} E_G \left[ \log_2 \left( 1 + \frac{\min\left\{ p_i^{sr} G, p_i^{rd} | h_{rd,i} |^2 \right\}}{N_0} \right) \right] \\
\text{s.t.} \quad T \sum_{i=1}^{m} p_i^{sr} \leq \sum_{i=1}^{m} E_{i-1}^s, \quad m = 1, 2, ..., N, \\
T \sum_{i=1}^{m} p_i^{rd} \leq \sum_{i=1}^{m} E_{i-1}^r, \quad m = 1, 2, ..., N, \\
p_i^{sr} E \left[ |h_{sr,i}|^2 \right] = p_i^{rd} E \left[ |h_{rd,i}|^2 \right], \\
P_{out} = P_{out}^s + P_{out}^d - P_{out}^{sr} - P_{out}^{rd} \leq \chi, \\
0 \leq p_i^{sr} \leq P_{max}, \quad 0 \leq p_i^{rd} \leq P_{max}. \quad (25)
\]

where the RV \( G \) is equal to \( \min\left\{ |h_{sr,i}|^2, |h_{rd,i}|^2 \right\} \) and its probability density function (PDF) and CDF is calculated in Appendix B. Since we are dealing with CDI, similar to the solution of the spectral efficiency maximization problem of a single-hop link, we find a closed-form solution for the problem for a specific and known value of \( G \).

The optimization problem in (23) is convex and can be solved by the dual problem (24). The Lagrangian associated to the primal problem can be calculated similar to the one for the single-hop problem in Appendix A. By calculating the first derivative of the Lagrangian function and applying KKT conditions and simplifying the result similar to the single-hop problem, the optimal allocated power to the source can be expressed as

\[
p_i^{sr} = \left[ \frac{\alpha}{T \sum_{j=1}^{N} \lambda_j} - \frac{N_0}{G} \right] P_{max} \left( 1 + \frac{1}{\psi} \right), \quad (26)
\]

where the value of \( \phi_i = \alpha \left( T \sum_{j=1}^{N} \lambda_j \right)^{-1} \) is water level.

2) Finite Battery Capacity: As it mentioned in Subsection III-A2, the summation of unused energy in the previous time slots in addition to the arrival energy in current time slot should be greater than the battery capacity. These constraints for source an relay is given by (27) and (28), respectively.

\[
\sum_{i=1}^{m+1} E_{i-1}^s - \sum_{i=1}^{m} T \frac{1}{2} p_i^{sr} \leq B_{max}^s, \quad m = 1, 2, ..., N - 1, \quad (27)
\]

\[
\sum_{i=1}^{m+1} E_{i-1}^r - \sum_{i=1}^{m} T \frac{1}{2} p_i^{rd} \leq B_{max}^r, \quad m = 1, 2, ..., N - 1, \quad (28)
\]

where \( B_{max}^s \) and \( B_{max}^r \) are the source and the relay capacity, respectively.

The throughput maximization problem with these new constraints can be rewritten as

\[
\max_{p_i^{sr}, p_i^{rd}} \sum_{i=1}^{N} \frac{1}{2} E_G \left[ \log_2 \left( 1 + \frac{\min\left\{ p_i^{sr} | h_{sr,i} |^2, p_i^{rd} | h_{rd,i} |^2 \right\}}{N_0} \right) \right] \\
\text{s.t.} \quad T \sum_{i=1}^{m} p_i^{sr} \leq \sum_{i=1}^{m} E_{i-1}^s, \quad m = 1, 2, ..., N, \\
T \sum_{i=1}^{m} p_i^{rd} \leq \sum_{i=1}^{m} E_{i-1}^r, \quad m = 1, 2, ..., N, \\
\sum_{i=1}^{m+1} E_{i-1}^s - \sum_{i=1}^{m} T \frac{1}{2} p_i^{sr} \leq B_{max}^s, \quad m = 1, 2, ..., N - 1, \\
\sum_{i=1}^{m+1} E_{i-1}^r - \sum_{i=1}^{m} T \frac{1}{2} p_i^{rd} \leq B_{max}^r, \quad m = 1, 2, ..., N - 1, \\
p_i^{sr} E \left[ |h_{sr,i}|^2 \right] = p_i^{rd} E \left[ |h_{rd,i}|^2 \right], \\
P_{out} = P_{out}^s + P_{out}^d - P_{out}^{sr} - P_{out}^{rd} \leq \chi, \\
0 \leq p_i^{sr} \leq P_{max}, \quad 0 \leq p_i^{rd} \leq P_{max}. \quad (29)
\]

As pointed out in Subsection III-B1, the problem in (29) can be simplified by the constraints of data causality and outage probability. Simplifying the constraints, (29) yields
\[
\ln p_i^{sr} \geq \ln (\beta N_0) - \frac{1}{\sigma_{sr} + \sigma_{rd}} \left[ \frac{2 (\mu_{sr} \sigma_{rd} + \mu_{rd} \sigma_{sr}) + \sigma_{sr} \ln \psi - \sigma_{sr} \sigma_{rd} \sqrt{8 \pi} (1 - \chi)}{\sigma_{sr} + \sigma_{rd}} \right]
\]

\[
p_i^{sr} \geq \beta N_0 \exp \left( \frac{-2 (\mu_{sr} \sigma_{rd} + \mu_{rd} \sigma_{sr}) + \sigma_{sr} \ln \psi - \sigma_{sr} \sigma_{rd} \sqrt{8 \pi} (1 - \chi)}{\sigma_{sr} + \sigma_{rd}} \right) = P_{\text{min}},
\]

(24)

\[
\max_{p_i^{sr}} \sum_{i=1}^{N} \frac{1}{2} E_G \left[ \log_2 \left( 1 + \frac{p_i^{sr} G}{N_0} \right) \right]
\]

s.t. \[
\frac{T}{2} \sum_{i=1}^{m} p_i^{sr} \leq \min \left( \sum_{i=1}^{m} E_{i-1}^s \sum_{i=1}^{m} E_{i-1}^r / \psi, \right),
\]

\[
\frac{T}{2} \sum_{i=1}^{m} p_i^{sr} \geq \max \left( \sum_{i=1}^{m+1} E_{i-1}^s \sum_{i=1}^{m+1} E_{i-1}^r / \psi \right),
\]

\[
P_{\text{min}} \leq p_i^{sr} \leq P_{\text{max}}.
\]

(30)

Since the optimization problem is convex, we find the closed-form solution by calculating the Lagrangian function, its derivative, and applying KKT conditions. The Lagrangian can be written similar to the single-hop finite battery problem described in Appendix A and by calculating its derivative the optimal value of the power is

\[
p_i^{sr} = \left[ \frac{\alpha}{T \left( \sum_{j=1}^{N} \lambda_j - \sum_{j=1}^{N-1} \nu_j \right)} - \frac{N_0}{G} \right] P_{\text{max}}, \quad (31)
\]

where in (31), \( \phi = \alpha \left[ T \left( \sum_{j=1}^{N} \lambda_j - \sum_{j=1}^{N-1} \nu_j \right) \right]^{-1} \) is water level in the \( i \)-th time slot.

IV. LOWER AND UPPER BOUNDS

In this section we focus on lower and upper bounds of the average spectral efficiency in the \( i \)-th time slot which is given as

\[
\bar{\eta}_{SE,i} = E \left[ \log_2 \left( 1 + \frac{|h_i|^2 p_i}{N_0} \right) \right]. \quad (32)
\]

We achieve a lower and an upper bounds of the optimization problems of single and dual-hop discussed in subsections III-A and III-B

A. Upper-Bound

According to the Jensen’s inequality, the upper bound for the objective function of the single-hop optimization problem given in (8) and (20) is

\[
E_h \left[ \log_2 \left( 1 + \frac{p_i |h_i|^2}{N_0} \right) \right] \leq \log_2 \left( 1 + \frac{p_i E_h [ |h_i|^2 ]}{N_0} \right).
\]

(33)

With this in mind, we can apply this inequality for the objective function of the dual-hop optimization problem. For the dual-hop problem, it is

\[
E_h \left[ \log_2 \left( 1 + \frac{p_i^{sr} |h_i|^2}{N_0} \right) \right] \leq \log_2 \left( 1 + \frac{p_i^{sr} E [ G ]}{N_0} \right), \quad (34)
\]

where \( G \) is the RV equals to the minimum gain of the both channels, i.e., \( \min \left\{ |h_{sr,i}|^2, \psi |h_{rd,i}|^2 \right\} \). In order to calculate the upper bound for the dual-hop problem, \( E [ G ] \) should be calculated and the details can be found in Appendix C. The final value for the expected value of the RV \( G \) is

\[
E [ G ] = \exp \left( 2 \mu_{sr} + 2 \sigma_{sr}^2 \right) \text{erfc} \left( \frac{\sqrt{\sigma_{sr}^2 + \sigma_{rd}^2}}{2} \right), \quad (35)
\]

where as mentioned before, \( \mu_{sr}, \sigma_{sr} \) ,and \( \sigma_{rd} \) are the channels parameters.

B. Lower-Bound

In (26), the function \( \log_2 (1 + ae^x) \) for \( a > 0 \) is suggested as a convex function. By applying Jensen’s inequality, the lower bound for the single-hop optimization problem in (8) is as

\[
\log_2 \left( 1 + \frac{p_i \exp \left( E [ |h_i|^2 ] \right)}{N_0} \right) \leq E_h \left[ \log_2 \left( 1 + \frac{p_i \exp \left( E [ |h_i|^2 ] \right)}{N_0} \right) \right].
\]

(36)

For the dual-hop problem, the inequality can be modified and it is necessary to calculate the expected value of the new RV \( Y \) as

\[
\log_2 \left( 1 + \frac{p_i^{sr} \exp \left( E [ Y ] \right)}{N_0} \right) \leq \cdots \leq E_h \left[ \log_2 \left( 1 + \frac{p_i^{sr} \exp \left( E [ Y ] \right)}{N_0} \right) \right],
\]

(37)

where \( Y = \ln \left( \min \left\{ |h_{sr,i}|^2, \psi |h_{rd,i}|^2 \right\} \right) \). In order to calculate the expected value of the RV \( Y \), the PDF of \( Y \) should be calculated, and the approach is similar to the procedure for the RV \( G \)'s calculation. The expected value of \( Y \) is calculated in the Appendix D is as follows.
Fig. 4: The average spectral efficiency for the third channel with infinite battery capacity in a single-hop communication, uniform and Poisson distribution EH.

Fig. 5: The optimal value, the upper bound, and the lower bound of the average spectral efficiency of the third channel with finite battery capacity in a single-hop communication.

Fig. 6: The average value of the optimal spectral efficiency for the second channel for the finite and infinite battery capacity in a dual-hop communication.

and the person had been walking freely around the room during the measurement [27]. In order to simulate the system model, for the single-hop link, three channels are considered. The transmitter sensors for the first and the second channel are placed on the head and the receiver nodes are located on the right hand and leg, respectively. The distance between the nodes in the first and second channel are 78cm and 105cm, respectively. For the third channel, the transmitter and the receiver are located on the chest and left leg, respectively and their distance is about 57cm. The simulation is run for $10^4$ iterations which channel coefficients is generated independently. According to the IEEE 802.15.6 standard, $P_{\text{max}}$ is assumed to be 0.1mW. The duration of each time slot, the channel bandwidth and the noise power are assumed to be 0.5sec, 10KHz and $-100$dB, respectively. The SNR threshold, $\beta$, and the outage probability threshold, $\chi$, are 10dB and 0.1%, respectively. For the cases that the battery capacity is finite, $B_{\text{max}}$ is supposed to be 0.1mJ.

For the dual-hop link, two channels are considered by using six sensors. In the first channel, the transmitter, the relay, and the receiver are located on the head, chest, and right hand, respectively which their distances are 41 and 46cm. The second channel includes three nodes which are placed on the head, the chest, and the right leg. According to the Section [4] the applied model is non-causal ESI and it is assumed that the harvested energy is known at the transmitter. The harvested energy is selected from the values of $E = [10, 20, \ldots, 90, 100]$ $\mu$J with uniform distribution in each time slot.

Fig. [4] depicts the average value of the spectral efficiency for the third channel which the transmitter and the receiver are located on the chest and left leg, respectively with infinite battery capacity. As it is shown and was expected, the average spectral efficiency with uniform EH is greater than Poisson distribution with the parameters $\lambda_e = 50\mu$J/s and $\lambda_e = 30\mu$J/s. It’s worth mentioning that the smaller the value of $\lambda_e$, the lower average spectral efficiency. The upper and lower bounds for the problem were calculated in the Section [V] Fig. [5] illustrates the average of the optimal spectral efficiency, the lower bound,
and the upper bound for the second channel. It is observed that the lower bound is a better approximation of the optimal value in comparison with the upper bound.

In Fig. 6 the average value of the optimal spectral efficiency in each time slot is illustrated. It compares the average spectral efficiency of a channel in a dual-hop communication for both the finite and infinite battery capacity. As it was expected, the system with infinite capacity has better performance than the finite one.

The comparison of the average spectral efficiency of a single and dual-hop is illustrated in Fig. 7. It is shown that the performance of the single-hop link is better than the dual-hop one. The result clearly shows that the relay degrades the performance. To put it in another way, the first channel in the dual-hop scenario is worse than the first channel in the single-hop one.

The energy consumption per bit for each channel of single and dual-hop links is depicted in Fig. 8. It compares the average spectral efficiency in comparison with the upper bound.

VI. Conclusion

A wireless body area network consists of some nodes on or in the body to monitor patient’s health. According to inaccessibility to these nodes, the energy provisioning is a pivotal issue in these networks. Harvesting energy from the environment in which these nodes are implanted is a possible solution for their energy supply. This paper studied the maximization of the spectral efficiency problem in an EH-aware wireless body area network over a block fading channels with the CDI knowledge at the transmitters. In this regard, we considered a single-hop link in a WBAN channel and formulate the optimization problem with some constraints on EH, maximum transmission power, and outage probability. This problem was solved for both finite and infinite battery capacities. In the next step, the dual-hop link which contains a relay node was considered as an extension and their performances were compared in the simulation results section. Inasmuch as the CDI is available to the transmitters, with the help of the Jensen’s inequality, an upper and lower bounds were calculated for the both system models.

APPENDIX A

Optimal Value of the Transmitting Power

A. Optimal Value of Single-Hop Infinite Battery

By defining the Lagrangian function and additional complimentary slackness conditions, we try to solve the optimization problem. For any \( \lambda_i \geq 0, \mu_i \geq 0 \) and \( \eta_i \geq 0 \), the Lagrangian function is as follows

\[
\mathcal{L} = \sum_{i=1}^{N} \log_2 \left( 1 + \frac{|h_i|^2 p_i}{N_0} \right) - \sum_{i=1}^{N} \lambda_i \left( \sum_{m=1}^{i} T p_m - \sum_{m=1}^{i-1} E_{m-1} \right) - \sum_{i=1}^{N} \mu_i (p_i - P_{\text{max}}) + \sum_{i=1}^{N} \eta_i (p_i - P_{\text{min}}). \tag{39}
\]

Corresponding complimentary slackness conditions are

\[
\lambda_i \left( \sum_{m=1}^{i} T p_m - \sum_{m=1}^{i-1} E_{m-1} \right) = 0, \forall i, \tag{40}
\]

\[
\mu_i (p_i - P_{\text{max}}) = 0, \forall i, \tag{41}
\]

\[
\eta_i (p_i - P_{\text{min}}) = 0, \forall i. \tag{42}
\]

By differentiating the Lagrangian with respect to \( p_i \), we obtain

\[
p_i^* = \left[ \frac{\alpha}{\mu_i - \eta_i + \sum_{j=1}^{N} \lambda_j} - \frac{N_0}{|p_i|^2} \right]. \tag{43}
\]

For the ease of exposure in (43), the value of \( 1/ \ln 2 \) is shown as \( \alpha \). Using complimentary slackness conditions expressed in (40), (41), (42), transmission power, \( p_i^* \), can be simplified more. As mentioned before, Lagrange multipliers \( \mu_i \) and \( \eta_i \)
are associated with $P_{\text{max}}$ and $P_{\text{min}}$ constraints, respectively. Now, we consider different cases of these multipliers to obtain the transmission power. According to the slackness conditions, $\mu_i$ and $\eta_i$ can not be zero, simultaneously. For the case that $\mu_i = 0$ and $\eta_i \neq 0$, $p_i^*$ can be simplified as $P_{\text{min}}$, in addition, for the case that $\mu_i \neq 0$ and $\eta_i = 0$, $p_i^*$ will be $P_{\text{max}}$ and for the last one which both of the multipliers $\mu_i$ and $\eta_i$ are zero, the optimal transmission power is obtained as [10].

B. Optimal Value of Single-Hop Finite Battery

For any $\lambda_i \geq 0$, $\mu_i \geq 0$, $\eta_i \geq 0$ and $\nu_i \geq 0$, the Lagrangian function is as follows

$$
\mathcal{L} = \sum_{i=1}^{N} \log_2 \left( 1 + \frac{|h_i|^2 p_i}{N_0} \right) - \sum_{i=1}^{N} \lambda_i \left( \sum_{m=1}^{i} T_{pm} - \sum_{m=1}^{i-1} E_{m-1} \right) - \sum_{i=1}^{N} \nu_i \left( \sum_{m=1}^{i} E_{m-1} - \sum_{m=1}^{i-1} T_{pm} - B_{\text{max}} \right) - \sum_{i=1}^{N} \mu_i (p_i - P_{\text{max}}) + \sum_{i=1}^{N} \eta_i (p_i - P_{\text{min}}) .
$$

(44)

In addition to the complimentary slackness conditions in the preceding problem, there is another slackness condition according to [11], that is a constraint on the battery capacity. By differentiating the Lagrangian with respect to $p_i$, allocated power for the $i$-th time slot is

$$
p_i^* = \left[ \frac{\alpha}{T \left( \sum_{j=i}^{N} \lambda_j - \sum_{j=i}^{N-1} \nu_j \right) + \mu_i - \eta_i \frac{|h_i|^2}{N_0}} \right].
$$

(45)

Similar to the previous optimization problem and its solution for the allocated power, this solution can be simplified by considering various cases of $\mu_i$ and $\eta_i$. The simplified $p_i^*$ is [13].

APPENDIX B

PDF OF THE RANDOM VARIABLE G

The RV $G$ is equal to the minimum of source-relay and relay-destination channel gains. With the help of CDF definition, we will calculate the PDF and CDF of the RV $G$ equaling $\min \{ |h_{sr,i}|^2, \psi |h_{rd,i}|^2 \}$. $|h_{sr,i}|^2$ and $\psi |h_{rd,i}|^2$ are distributed log-normally with parameters $(2\mu_{sr}, 4\sigma_{sr}^2)$ and $(2\mu_{rd} + 2 (\sigma_{sr}^2 - \sigma_{rd}^2), 4\sigma_{rd}^2)$, respectively.

$$
\mathcal{F}_G(g) = \Pr( G \leq g ) = \Pr \left( \min \left\{ |h_{sr,i}|^2, \psi |h_{rd,i}|^2 \right\} \leq g \right) = 1 - \Pr \left( |h_{sr,i}|^2 \geq g \right) \Pr \left( \psi |h_{rd,i}|^2 \geq g \right),
$$

(46)

By substituting the corresponding probability distributions, (46) can be rewritten as

$$
\mathcal{F}_G(g) = 1 - \frac{1}{4} \left\{ 1 - \text{erf} \left( \frac{\ln g - 2\mu_{sr}}{2\sigma_{sr}\sqrt{2}} \right) \right\} \cdot \left[ 1 - \text{erf} \left( \frac{\ln g - 2\mu_{sr} - 2 (\sigma_{sr}^2 - \sigma_{rd}^2)}{2\sigma_{rd}\sqrt{2}} \right) \right].
$$

(47)

Now by calculating the first derivative of the CDF, the PDF is achieved as

$$
f_G(g) = \frac{\partial \mathcal{F}_G(g)}{\partial g} = \frac{1}{4\sqrt{2\pi}g} \left[ \frac{1}{\sigma_{sr}} \exp \left( -\frac{\ln g - 2\mu_{sr}}{8\sigma_{sr}^2} \right) - \frac{1}{\sigma_{sr}} \exp \left( -\frac{\ln g - 2\mu_{sr} - 2 (\sigma_{sr}^2 - \sigma_{rd}^2)}{8\sigma_{rd}^2} \right) \right] \cdot \left[ 1 - \text{erf} \left( \frac{\ln g - 2\mu_{sr} - 2 (\sigma_{sr}^2 - \sigma_{rd}^2)}{2\sigma_{rd}\sqrt{2}} \right) \right].
$$

(48)

where erfc(.) is complementary error function and defined as $\text{erfc}(x) = 1 - \text{erf}(x)$.

APPENDIX C

THE EXPECTED VALUE OF G

In order to calculate the expected value of the RV $G$, we start with the definition of expectation.

$$
\mathbb{E}[G] = \int_{0}^{\infty} g f_G(g) \, dg .
$$

(49)

Now, by substituting (48) into (49) and changing the variable $g$ and replacing with $e^x$, the rewritten version of the integral is

$$
\mathbb{E}[G] = \frac{1}{4\sqrt{2\pi}} \int_{-\infty}^{\infty} e^x \left[ \frac{1}{\sigma_{sr}} \exp \left( -\frac{(x - 2\mu_{sr})^2}{8\sigma_{sr}^2} \right) + \cdots \right. \\
+ \frac{1}{\sigma_{rd}} \exp \left( -\frac{(x - 2\mu_{sr} - 2 (\sigma_{sr}^2 - \sigma_{rd}^2))^2}{8\sigma_{rd}^2} \right) \\
- \frac{1}{\sigma_{sr}} \exp \left( -\frac{(x - 2\mu_{sr} - 2 (\sigma_{sr}^2 - \sigma_{rd}^2))^2}{8\sigma_{rd}^2} \right) \text{erf} \left( \frac{x - 2\mu_{sr} - 2 (\sigma_{sr}^2 - \sigma_{rd}^2)}{2\sigma_{rd}\sqrt{2}} \right) \\
- \frac{1}{\sigma_{rd}} \exp \left( -\frac{(x - 2\mu_{sr} - 2 (\sigma_{sr}^2 - \sigma_{rd}^2))^2}{8\sigma_{rd}^2} \right) \text{erfc} \left( \frac{x - 2\mu_{sr} - 2 (\sigma_{sr}^2 - \sigma_{rd}^2)}{2\sigma_{rd}\sqrt{2}} \right) \] dx.
$$

(50)

In order to calculate the integral in (50), we need to calculate the integral contains error function and a Gaussian function. It is proved in Appendix E, and it’s corollary is

$$
\int_{-\infty}^{\infty} e^{-(\alpha x + \beta)^2} \text{erf} \left( \gamma x + \delta \right) dx = \frac{\sqrt{\pi}}{\alpha} \text{erf} \left( \frac{\alpha \delta - \beta \gamma}{\sqrt{\alpha^2 + \gamma^2}} \right).
$$

(51)
Therefore, by calculating the integral with the help of (51), (50) simplifies to
\[
\mathbb{E}[G] = \exp(2\mu_s + 2\sigma_{sr}^2) \left[ 1 - \text{erf}\left( \frac{\sqrt{\sigma_{sr}^2 + \sigma_{rd}^2}}{2} \right) \right] = \exp(2\mu_s + 2\sigma_{sr}^2) \text{erfc}\left( \frac{\sqrt{\sigma_{sr}^2 + \sigma_{rd}^2}}{2} \right). \quad (52)
\]

**APPENDIX D**

**THE EXPECTED VALUE OF Y**

In order to achieve the average value of the RV \( Y \), by definition
\[
\mathbb{E}[Y] = \int_{-\infty}^{+\infty} y \frac{1}{4\pi^2} \exp\left(-\frac{(y - 2\mu_s)^2}{8\sigma_{sr}^2} \right) \, dy + \frac{1}{\sigma_{rd}} \exp\left(-\frac{(y - 2\mu_s - 2(\sigma_{sr}^2 - \sigma_{rd}^2))^2}{8\sigma_{rd}^2} \right) \, dy - \frac{1}{\sigma_{sr}} \exp\left(-\frac{(y - 2\mu_s - 2(\sigma_{sr}^2 - \sigma_{rd}^2))^2}{8\sigma_{sr}^2} \right) \, dy - \frac{1}{\sigma_{rd}} \exp\left(-\frac{(y - 2\mu_s - 2(\sigma_{sr}^2 - \sigma_{rd}^2))^2}{8\sigma_{rd}^2} \right) \, dy. \quad (53)
\]

Before focusing on the integral solution, in order to calculate the integral in (53), the first derivative of the equation (51) with respect to the parameter \( \alpha \) is calculated.
\[
\int_{-\infty}^{+\infty} x e^{-(\alpha x + \beta)^2} \text{erf}(\gamma x + \delta) \, dx = \cdots
\]
\[
= \frac{\gamma}{\alpha^2 \sqrt{\alpha^2 + \gamma^2}} \exp\left(-\frac{(\alpha \delta - \beta \gamma)^2}{\alpha^2 + \gamma^2} \right) - \frac{\beta \sqrt{\pi}}{\alpha^2} \text{erf}\left( \frac{\alpha \delta - \beta \gamma}{\sqrt{\alpha^2 + \gamma^2}} \right). \quad (54)
\]

Therefore, the average value of the RV \( Y \) given in (38) can be easily deduced.

**APPENDIX E**

**PROOF OF THE INTEGRAL CONTAINING \( e^{-x^2} \) AND \( \text{erf}(x) \)**

\[
\int_{-\infty}^{+\infty} e^{-(\alpha x + \beta)^2} \text{erf}(\gamma x + \delta) \, dx = \frac{\sqrt{\pi}}{\alpha} \text{erf}\left( \frac{\alpha \delta - \beta \gamma}{\sqrt{\alpha^2 + \gamma^2}} \right). \quad (55)
\]

We define \( \mathcal{I}(\delta) \) as follows
\[
\mathcal{I}(\delta) = \int_{-\infty}^{+\infty} e^{-(\alpha x + \beta)^2} \text{erf}(\gamma x + \delta) \, dx
\]
\[
\Rightarrow \frac{\partial \mathcal{I}(\delta)}{\partial \delta} = \frac{2}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-(\alpha x + \beta)^2 - (\gamma x + \delta)^2} \, dx. \quad (56)
\]

Now, by manipulating (56), we make the \( \exp(.) \)’s exponent a perfect square.

\[
\frac{\partial \mathcal{I}(\delta)}{\partial \delta} = \frac{2}{\sqrt{\pi}} \exp\left(-\frac{(\alpha \delta - \beta \gamma)^2}{\alpha^2 + \gamma^2} \right) \\
\int_{-\infty}^{+\infty} \exp\left[-\left(\frac{(\alpha^2 + \gamma^2)^{1/2} x + \frac{\alpha \beta + \delta \gamma}{(\alpha^2 + \gamma^2)^{1/2}}\right)^2\right] dx. \quad (57)
\]

For ease of exposure, we define \( A \) and \( B \) as
\[
A = -(\alpha^2 + \gamma^2)^{1/2} x + \frac{\alpha \beta + \delta \gamma}{(\alpha^2 + \gamma^2)^{1/2}}, \quad B = -\frac{(\alpha \delta - \beta \gamma)^2}{\alpha^2 + \gamma^2}.
\]

The answer of the integral term in (57) is blow.
\[
\int_{-\infty}^{+\infty} \exp\left(\frac{\sqrt{\pi}}{2(\alpha^2 + \gamma^2)^{1/2}} \text{erf}(A) \right) \, dx \quad (58)
\]

Therefore, we have
\[
\frac{\partial \mathcal{I}(\delta)}{\partial \delta} = \frac{1}{(\alpha^2 + \gamma^2)^{1/2}} \text{exp}(B) \text{erf}(A) \quad (59)
\]

In order to calculate \( \mathcal{I}(\delta) \), we make use of definite integral. Therefore,
\[
\mathcal{I}(\delta) - \mathcal{I}(-\infty) = \int_{-\infty}^{\delta} \frac{2}{(\alpha^2 + \gamma^2)^{1/2}} \exp\left(-\frac{(\alpha t - \beta \gamma)^2}{\alpha^2 + \gamma^2} \right) dt.
\]
\[
\mathcal{I}(\delta) = \frac{\sqrt{\pi}}{\alpha} \text{erf}\left( \frac{\alpha \delta - \beta \gamma}{\sqrt{\alpha^2 + \gamma^2}} \right). \quad (60)
\]

The value of the first term on the right hand of (61) is \( \frac{\sqrt{\pi}}{\alpha} \), and the second term is the definition of error function. Therefore, it simplifies to
\[
\mathcal{I}(\delta) = \frac{\sqrt{\pi}}{\alpha} \text{erf}\left( \frac{\alpha \delta - \beta \gamma}{\sqrt{\alpha^2 + \gamma^2}} \right). \quad (62)
\]

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